

ON THE LIMIT SET OF ANOSOV REPRESENTATIONS

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ABSTRACT. We study the limit set of discrete subgroups arising from Anosov representations. Specially we study the limit set of discrete groups arising from strictly convex real projective structures and Anosov representations from a finitely generated word hyperbolic group into a semisimple Lie group.

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1. INTRODUCTION

There has been intensive study about the limit set of rank one symmetric spaces. Nonetheless it is still mysterious how the limit set of higher rank symmetric spaces looks like. In [13], it is analyzed that some Tits neighborhoods of parabolic fixed points of nonuniform lattices in higher rank semisimple Lie groups do not include conical limit points, which is a sharp contrast to real rank one case. In this paper we try to describe some examples and results related to the linear action and a geometric structure which arise as a discrete subgroup of $SL(d, \mathbb{R})$. This example naturally arises as convex projective structures on surfaces. More generally such groups appear in so-called Hitchin component of representation variety of surface group in $SL(d, \mathbb{R})$.

A strictly convex real projective structure on surfaces is a generalization of hyperbolic structure. Nonetheless if we look at the action on $SL(3, \mathbb{R})/SO(3)$ instead of on \mathbb{RP}^2 , it is not obvious that we can get the same phenomena as in $SL(2, \mathbb{R})/SO(2)$. Yet it shares many parallel properties since the Hilbert metric associated to the projective structure is more or less hyperbolic like. This is our motivation to study limit sets in $SL(3, \mathbb{R})/SO(3)$ arising from such a geometric structure and attempt to classify the limit points. Another motivation is to compare the action on $SL(3, \mathbb{R})/SO(3)$ and the natural linear action on \mathbb{R}^3 . The latter relation will be investigated in a future paper.

We begin by defining types of limit points on the geometric boundary of general symmetric spaces. The notions of radial limit point and horospherical limit point are introduced by Albuquerque in [2] and Hattori in [13] as in the theory of Kleinian groups.

Definition 1.1 ([13]). Let Γ be a discrete subgroup of $\text{Isom}(X)$ where X is a symmetric space of noncompact type. A limit point $\xi \in \partial X$ is *horospherical* if there exists a sequence $\gamma_n \in \Gamma$ so that for any horoball B based at ξ , $\gamma_n o$ is contained in B for all large n .

Definition 1.2 ([2]). A limit point $\xi \in \partial X$ is called a *radial limit point* if there exists a sequence $\gamma_n \in \Gamma$ such that $\gamma_n o$ converges to ξ in the cone topology and remains at a bounded distance of the union of closed Weyl chambers with apex o containing the geodesic ray $\sigma_{o, \xi}$.

The notion of conical limit point is also defined in [13]. The condition is stronger than being a radial limit point. It is easily seen that every

limit point for a uniform lattice is radial. Hattori [13] characterizes exactly radial (conical) limit points for \mathbb{Q} -rank 1 lattices. He also shows that every limit point for a finitely generated generalized Schottky group in $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ is horospherical. It seems to be difficult to classify limit points of general discrete subgroup of higher rank symmetric space. In this paper, we prove that

Theorem 1.3. *Let Γ be a discrete subgroup of $\mathrm{SL}(3, \mathbb{R})$ arising from a convex real projective structure on a closed surface. Let X be the symmetric space associated to $\mathrm{SL}(3, \mathbb{R})$. Then the limit set Λ_Γ of Γ in the Furstenberg boundary of X is homeomorphic to S^1 and the limit set L_Γ in the geometric boundary of X splits as a product $S^1 \times I$ where I is the closed interval identified with the directions of the limit cone.*

In addition to Theorem 1.3, we characterize radial limit points of Γ in the geometric boundary of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$.

Theorem 1.4. *Let Γ be a discrete subgroup of $\mathrm{SL}(3, \mathbb{R})$ arising from a convex real projective structure on a closed surface. Every limit point of Γ is horospherical. Furthermore there is only one radial limit point in each Weyl chamber at infinity with nonempty limit set of Γ .*

It is well known that the Hitchin component of the representation variety of a surface group in $\mathrm{SL}(3, \mathbb{R})$ is equal to the deformation space of convex projective structures on the surface [6]. Due to Theorem 1.3 and 1.4, one can see how the structure of limit set is changed in the Hitchin component for $\mathrm{SL}(3, \mathbb{R})$ as follows: Let Γ_0 be a discrete subgroup of $\mathrm{SL}(3, \mathbb{R})$ arising from a hyperbolic structure on a closed surface. Then it is a standard fact that the limit set of Γ_0 in the geometric boundary of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ is homeomorphic to a circle S^1 and the limit set in any Weyl chamber at infinity, if nonempty, consists of a point. Moreover, it can be easily seen that every limit point of Γ_0 is a radial limit point.

When Γ_0 is deformed to a discrete subgroup of $\mathrm{SL}(3, \mathbb{R})$ arising from a convex real projective structure on the surface, the limit set in each Weyl chamber at infinity with nonempty limit set is changed from a point to an interval and thus limit set is changed from a circle to a cylinder. Even though the limit set in each Weyl chamber at infinity suddenly increases from a point to an interval, it turns out due to Theorem 1.4 that the set of radial limit points in each Weyl chamber

at infinity does not increase. Indeed, there exists only one point in each interval which is a radial limit point and hence, the number of radial limit points in each Weyl chamber at infinity is preserved under the deformation of Γ_0 . To our knowledge, this is the first example of a concrete description of limit set in higher rank symmetric space, except for limit sets of lattices.

All the machinery to show the above theorems work equally well for any Anosov representations, see section 8. Hence we have

Theorem 1.5. *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense discrete P -Anosov representation from a word hyperbolic group Γ where P is a minimal parabolic subgroup of a semisimple Lie group G . Then the geometric limit set is isomorphic to the set $\partial\Gamma \times \partial\mathcal{L}_{\rho(\Gamma)}$. Furthermore in each Weyl chamber intersecting the geometric limit set nontrivially, there is only one radial limit point.*

Here $\partial\mathcal{L}_{\rho(\Gamma)}$ is the set of directions of limit cone $\mathcal{L}_{\rho(\Gamma)}$. See section 3 for definitions.

2. PRELIMINARIES

Let G be a higher rank semisimple real Lie group and X an associated symmetric space. For each $\xi \in \partial X$, there is an associated parabolic group P_ξ which is a stabilizer of ξ in G . Then P_ξ has a generalized Iwasawa decomposition $P_\xi = N_\xi A_\xi K_\xi$, where K_ξ is a subgroup of an isotropy group of a fixed point o in X , $A_\xi o$ is the union of parallels to a geodesic l connecting o and ξ , and N_ξ is the horospherical subgroup determined only by ξ . If ξ is a regular point, P_ξ becomes a minimal parabolic subgroup. In this case the group G has a Iwasawa decomposition $G = KAN$, where K is an isotropy group of o , Ao is a maximal flat, and N is a nilpotent group stabilizing the regular point ξ . A choice of a Weyl chamber \mathfrak{a}^+ in \mathfrak{a} , the Lie algebra of A , determines a positive root and accordingly a fundamental system Υ of roots. $A^+\xi$ is also called a Weyl chamber with an apex ξ where $A^+ = \exp(\mathfrak{a}^+)$. A choice of a subset $\Theta \subset \Upsilon$ determines a face of \mathfrak{a}^+

$$\mathfrak{a}^\Theta = \{H \in \overline{\mathfrak{a}^+} \mid \alpha(H) = 0, \alpha \in \Theta\}.$$

So any singular element $\xi \in \partial X$ can be represented by an element in \mathfrak{a}^Θ for some $\Theta \subset \Upsilon$.

If ξ is a singular point, K_ξ is a centraliser of \mathfrak{a}^Θ in K , $A \subset A_\xi$, and $N_\xi \subset N$. In this case

$$G = KA_\xi N_\xi$$

is called a generalised Iwasawa decomposition. See [8] for details.

In terms of Lie algebras, we can describe parabolic subgroups as follows. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subset as before. The adjoint action of \mathfrak{a} gives rise to a decomposition of \mathfrak{g} into eigenspaces

$$\mathfrak{g} = \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha, \text{ where } \mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [a, x] = \alpha(a)x \text{ for } \forall a \in \mathfrak{a}\}.$$

Here Σ is the system of restricted roots of \mathfrak{g} . Let $N_K(\mathfrak{a})$ and $Z_K(\mathfrak{a})$ be the normalizer and the centralizer of \mathfrak{a} in K . The Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ acts on \mathfrak{a} and on Σ . A unique element $\omega_{op} \in W$ sending Σ^- to Σ^+ induces an involution

$$\iota : \Sigma^+ \rightarrow \Sigma^+, \alpha \mapsto -\omega_{op}(\alpha),$$

called the opposite involution $\iota(\Upsilon) = \Upsilon$.

The subalgebra

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$

is nilpotent and $N = \exp(\mathfrak{n}^+)$ is unipotent. The subgroup $B = Z_K(\mathfrak{a})AN$ is a minimal parabolic subgroup with its Lie algebra $\mathfrak{b}^+ = \mathfrak{g}_0 \oplus \mathfrak{n}^+$. Similarly one can define N^-, B^- using negative roots. The group B^- is conjugate to B^+ . In general, parabolic subgroups of G are conjugate to subgroups containing B^+ . A pair of parabolic subgroups is opposite if their intersection is a reductive group. The conjugacy classes of parabolic subgroups are in one to one correspondence with subsets $\Theta \subset \Upsilon$. For each Θ , let $\mathfrak{a}_\Theta = \bigcap_{\alpha \in \Theta} \ker \alpha$ and $M_\Theta = Z_K(\mathfrak{a}_\Theta)$ its centralizer in K . Then

$$P_\Theta^+ = M_\Theta AN \text{ and } P_\Theta^- = M_\Theta AN^-$$

are opposite parabolic subgroups. Any pair of opposite parabolic subgroup is conjugate to (P_Θ^+, P_Θ^-) for some $\Theta \subset \Upsilon$. The intersection $L_\Theta = P_\Theta^+ \cap P_\Theta^-$ is the common Levi component of P_Θ^+ and P_Θ^- . The group M_Θ is a maximal compact subgroup of L_Θ .

Note that P_Θ^- is conjugate to $P_{\iota(\Theta)}^+$. In particular P_Θ^+ is conjugate to its opposite if and only if $\Theta = \iota(\Theta)$. In our case, we will deal with

minimal parabolic subgroups B^+, B^- , hence they are opposite and conjugate.

A *geometric boundary* (or ideal boundary) ∂X of X is defined as the set of equivalence classes of geodesic rays under the equivalence relation that two rays are equivalent if they are within finite Hausdorff distance of each other. For any point $x \in X$ and any ideal point $\xi \in \partial X$, there exists a unique unit speed ray starting from x which represents ξ . The pointed Hausdorff topology on rays emanating from $x \in X$ induces a topology on ∂X . This topology does not depend on the base point x and is called the *cone topology* on ∂X .

Example. Let $G = \mathrm{SL}(d, \mathbb{R})$ and \mathfrak{g} its Lie algebra, the set of traceless (d, d) -matrices. The inner product $\langle Y, Z \rangle = \mathrm{Tr}(YZ^t)$ is a positive definite inner product on \mathfrak{g} which is a usual inner product on \mathbb{R}^{d^2} . The associated symmetric space X can be identified with the set of positive definite symmetric matrices with determinant 1, and $\mathrm{SL}(d, \mathbb{R})$ acts on it by conjugation $x \rightarrow gxg^t$. The isotropy group of the identity matrix $I \in \mathrm{SL}(d, \mathbb{R})$ is $\mathrm{SO}(d)$, hence $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$. We will denote o the class of I in $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$. If \mathfrak{k} denotes the Lie algebra of $\mathrm{SO}(d)$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition where \mathfrak{p} is identified with $T_o X$. Furthermore,

$$\mathfrak{a} = \left\{ \mathrm{Diag}(\lambda_1, \lambda_2, \dots, \lambda_d) \mid \sum_{i=1}^d \lambda_i = 0 \right\}$$

is a maximal abelian subspace of \mathfrak{p} and we choose

$$\mathfrak{a}^+ = \left\{ \mathrm{Diag}(\lambda_1, \lambda_2, \dots, \lambda_d) \mid \sum_{i=1}^d \lambda_i = 0, \lambda_1 > \lambda_2 > \dots > \lambda_d \right\}.$$

For $Y \in \mathfrak{p}$ with $\|Y\| = 1$ we denote by σ_Y the unique (unit speed) geodesic such that $\sigma_Y(0) = o$ and $\sigma'_Y(0) = Y$. In particular $\sigma_Y(s) = \exp(Ys) \cdot o$. Any point $\xi \in \partial X$ is realised as $\sigma_Y(\infty)$ for some $Y \in \mathfrak{p}$ with $\|Y\| = 1$. Let $\lambda_1(Y) > \dots > \lambda_k(Y)$ be distinct eigenvalues of Y and $E_i(Y)$ be the eigenspace of Y corresponding to $\lambda_i(Y)$. Set $V_i(Y) = \bigoplus_{j=1}^i E_j(Y)$. Then we get a flag

$$V_1(Y) \subset \dots \subset V_k(Y) = \mathbb{R}^d.$$

If m_i is the dimension of $E_i(Y)$, the following two conditions

$$\begin{aligned} \text{(i)} \quad & \sum_{i=1}^k m_i \lambda_i(Y) = 0 \\ \text{(ii)} \quad & \sum_{i=1}^k m_i \lambda_i(Y)^2 = 1 \end{aligned}$$

are satisfied due to the fact that Y is traceless and a unit vector.

In this way one gets a one-to-one correspondence between ∂X and the set of flags with two conditions. If $F(Y)$ is a flag associated with a point in ∂X , the action of $g \in \mathrm{SL}(d, \mathbb{R})$ is just $gF(Y)$. The typical example is when $Y \in \mathfrak{a}$ is a diagonal matrix with distinct entries. The corresponding flag to \mathfrak{a}^+ is just

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \dots, e_{d-1} \rangle \subset \mathbb{R}^d.$$

Changing eigenvalues corresponds to moving around in the same Weyl chamber. The adjacent Weyl chamber with $\lambda_2 > \lambda_1 > \lambda_3 > \cdots > \lambda_d$ is the flag

$$\langle e_2 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \dots, e_{d-1} \rangle \subset \mathbb{R}^d$$

and the opposite Weyl chamber with $\lambda_d > \lambda_{d-1} > \cdots > \lambda_1$ is

$$\langle e_d \rangle \subset \langle e_d, e_{d-1} \rangle \subset \cdots \subset \mathbb{R}^d.$$

Note that two Weyl chambers corresponding to two flags $V_1 \subset \cdots \subset V_d$ and $W_1 \subset \cdots \subset W_d$ are opposite if for any $i + j = d$,

$$(1) \quad V_i \oplus W_j = \mathbb{R}^d$$

We refer the reader to [8, Section 2.13] for more details about this.

In our case, it is particularly interesting when ξ is a singular point. Let $H_1 := \sqrt{(d-1)/d} \operatorname{Diag}(1, -1/(d-1), \dots, -1/(d-1)) \in \overline{\mathfrak{a}_1^+}$ be a diagonal matrix with last $d-1$ entries the same. This vector in \mathfrak{p} denotes a (maximal) singular direction by a geodesic σ_{H_1} starting from o and ending at a point $\xi_1 \in \partial X$ which we will denote by ∞ . Let r be a singular geodesic ray which is the image of σ_{H_1} and $\mathrm{SL}(d, \mathbb{R}) = \mathrm{SO}(d)A_\infty N_\infty$ a generalised Iwasawa decomposition. There is a nice description of the set A_∞ and N_∞ .

- (1) $g \in N_\infty$ if and only if (i) $g_{ij} = 0$ whenever $\lambda_j \geq \lambda_i$ and (ii) $g_{jj} = 1$. So N_∞ are upper triangular with 1's on the diagonal.

- (2) $g \in A_\infty$ if and only if $g_{ij} = 0$ whenever $\lambda_i \neq \lambda_j$ and g is symmetric positive definite.

It is not difficult to see that the union of parallels to r is $A_\infty \cdot o$ and is isometric to $\mathbb{R} \times \mathrm{SL}(d-1, \mathbb{R})/\mathrm{SO}(d-1)$, see [15]. Note here that the \mathbb{R} -factor is exactly the singular geodesic r . The orbit $N_\infty I$ is perpendicular to this set. A level set of a Busemann function centered at ∞ is $r(t_0) \times \mathrm{SL}(d-1, \mathbb{R})/\mathrm{SO}(d-1)$ together with $N_\infty \cdot (r(t_0) \times \mathrm{SL}(d-1, \mathbb{R})/\mathrm{SO}(d-1))$. In matrix form an element of A_∞ looks like

$$\begin{bmatrix} \mu & 0 \\ 0 & M \end{bmatrix}$$

where $\mu > 0$, and M is a positive definite symmetric $(d-1, d-1)$ -matrix with determinant equal to $1/\mu$. An element of K_∞ also looks like

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & M' \end{bmatrix}$$

where $M' \in \mathrm{O}(d-1)$.

Now, we recall the definition of the limit set of a discrete group.

Definition 2.1. Let Γ be a discrete subgroup of G . The (*geometric*) *limit set* L_Γ of Γ is defined by $L_\Gamma = \overline{\Gamma \cdot o} \cap \partial X$.

We remark that this definition does not depend on the chosen base point o and can be extended to isometry groups of arbitrary Hadamard manifolds. We call each point of L_Γ a *limit point* of Γ .

Definition 2.2. Let $\sigma : [0, \infty) \rightarrow X$ be a geodesic ray. The Busemann function $b(\sigma) : X \rightarrow \mathbb{R}$ associated with σ is given by

$$b(\sigma)(x) = \lim_{t \rightarrow \infty} (d(x, \sigma(t)) - t) \text{ for } x \in X.$$

For any real number C , we call the set $b(\sigma)^{-1}((-\infty, C))$ an open *horoball* centered at $\sigma(\infty)$, and the set $b(\sigma)^{-1}(C)$ a *horosphere* centered at $\sigma(\infty)$.

It is not easy to classify limit points for arbitrary discrete groups of higher rank symmetric space but we here give an example for which every limit point is a radial limit point as follows.

Example. Let $Y = X_1 \times X_2$ be a product of \mathbb{R} -rank one symmetric spaces. Let $\Gamma \subset \mathrm{Isom}^+(X_1) \times \mathrm{Isom}^+(X_2)$ be a group acting freely on

Y. Set $\Gamma_1 \subset \text{Isom}^+(X_1)$ be the projection of Γ to the first factor. If the geometric limit set L_Γ consists of only regular points, then by [7], $\Gamma = \{(\gamma, \phi\gamma) \mid \gamma \in \Gamma_1\}$ is a graph group for some type preserving isomorphism ϕ . Furthermore $L_\Gamma = \Lambda_\Gamma \times [a, b]$ where Λ_Γ is a limit set in Furstenberg boundary and

$$[a, b] = \overline{\left\{ \frac{l(\gamma)}{l(\phi\gamma)} \mid \gamma \in \Gamma_1 \text{ is hyperbolic} \right\}} \subset \mathbb{R}$$

is a closed interval. Here $l(\gamma)$ denotes the translation length of γ . Furthermore for any $[(\xi_1, \xi_2), p] \in L_\Gamma$, there is a sequence of hyperbolic isometries $\{(\gamma_i^1, \gamma_i^2)\}$ so that

$$(\gamma_i^1)^+ \rightarrow \xi_1, (\gamma_i^2)^+ \rightarrow \xi_2, \frac{l(\gamma_i^1)}{l(\gamma_i^2)} \rightarrow p$$

where γ^+ denotes the attractive fixed point of γ i.e., $\gamma^+ = \lim_{j \rightarrow \infty} \gamma^j o$. This implies that the Weyl chambers determined by $(\gamma_i^1)^+$ and $(\gamma_i^2)^+$ converges to the Weyl chamber determined by ξ_1 and ξ_2 , and the slope of the invariant axis of (γ_i^1, γ_i^2) converge to p . Then it is not difficult to show that $[(\xi_1, \xi_2), p]$ is a radial limit point.

3. LIMIT CONE, JORDAN DECOMPOSITION AND CARTAN DECOMPOSITION

An element g of a real reductive connected linear group can be uniquely written

$$g = ehu$$

where e is elliptic (all its complex eigenvalues have modulus 1), h is hyperbolic (all the eigenvalues are real and positive) and u is unipotent ($u - I$ is nilpotent), and all three commute [14]. This decomposition is called the Jordan decomposition of g . If $G = KAN$ is any Iwasawa decomposition of a semisimple Lie group G , e is conjugate to an element in K , h is conjugate to an element in A , and u is conjugate to an element in N [14]. The *translation length* $l(\alpha)$ of an isometry α is defined by

$$\inf_{x \in G/K} d(x, \alpha(x)).$$

It is shown in [8] that when G is a real semisimple Lie group, for $g \in G$, if $g = ehu$ is the Jordan decomposition, then $l(g) = l(h)$.

Fix a closed Weyl chamber $A^+ \subset G$ and denote $\lambda : G \rightarrow A^+$ the natural projection induced from the Jordan decomposition: for $g \in G$, $\lambda(g)$ is a unique element in A^+ which is conjugate to the hyperbolic component h of $g = ehu$. Note that $\lambda(g^n) = \lambda(g)^n$ since $g^n = e^n h^n u^n$. For $g \in \mathrm{SL}(n, \mathbb{R})$, $\log \lambda(g)$ is the vector in \mathfrak{a}^+ whose coordinates are logarithms of the absolute values of eigenvalues of g arranged in a decreasing order. Since $l(\lambda(g)) = |\log(\lambda(g))|$, we get the following corollary.

Corollary 3.1. *Let G be a real semisimple Lie group, and $g = ehu$ in G in its Jordan decomposition. Then*

$$|\log(\lambda(g))| = l(\lambda(g)) = l(h) = l(g).$$

Let ω_{op} be the element in the Weyl group of \mathfrak{a} which maps \mathfrak{a}^+ to $-\mathfrak{a}^+$. The opposite involution $\iota : \mathfrak{a}^+ \rightarrow \mathfrak{a}^+$ is defined to be: for $X \in \mathfrak{a}^+$, $\iota(X) = \mathrm{Ad}_{\omega_{op}}(-X)$. The *limit cone* \mathcal{L}_Γ of Γ is the smallest closed cone in \mathfrak{a}^+ containing $\log(\lambda(\Gamma))$. Benoist [3] showed

Theorem 3.2. *Let G be a real linear connected semisimple Lie group. If Γ is a Zariski-dense sub-semigroup of G , then the limit cone is convex and its interior is nonempty. If Γ is a Zariski dense subgroup, then \mathcal{L}_Γ is invariant under the opposite involution ι . Moreover the limit set of Γ in any Weyl chamber at infinity, if nonempty, is naturally identified with the set of directions in \mathcal{L}_Γ .*

4. WELL-DISPLACING REPRESENTATIONS AND U-PROPERTY

Let γ be an isometry of a metric space Y . We recall that the translation length of γ is $d_Y(\gamma) = \inf_{x \in Y} d(x, \gamma(x))$. We observe that $d_Y(\gamma)$ is an invariant of the conjugacy class of γ . If C_Γ is the Cayley graph of a group Γ with set of generators S and word length $\|\cdot\|_S$, the displacement function is called the translation length and is denoted by ℓ_S

$$\ell_S(\gamma) = \inf_{\eta} \|\eta \gamma \eta^{-1}\|_S.$$

Note that this is equal to the number of generators involved to write γ in a cyclically reduced way. We finally say the action by isometries on X of a group Γ is *well-displacing*, if given a set S of generators of Γ , there exist positive constants A and B such that

$$d_Y(\gamma) \geq A\ell_S(\gamma) - B.$$

This definition does not depend on the choice of S . From the definition, it is immediate that for $\rho : \Gamma \rightarrow \text{Isom}(Y)$ to be a well-displacing representation, it must be discrete and faithful, and the image consists of only hyperbolic isometries.

For hyperbolic groups, well-displacing action is equivalent to that the orbit map is a quasi-isometric embedding from the Cayley graph C_Γ to Y [11], i.e, for any $x \in Y$, there exist constants A and B so that

$$A^{-1}\|\gamma\| - B \leq d(x, \gamma(x)) \leq A\|\gamma\| + B.$$

We say that a finitely generated group has *U-property* if there exists finitely many elements g_1, \dots, g_p of Γ , positive constants A and B such that for any $\gamma \in \Gamma$,

$$\|\gamma\| \leq A \sup_i \ell(g_i \gamma) + B.$$

It is shown [11] that a closed surface group has *U-property*. If a representation of a group with *U-property* is well-displacing, then

$$\begin{aligned} d(x, \gamma x) &\geq \sup d(g_i^{-1}x, \gamma x) - \sup d(x, g_i^{-1}x) \\ &\geq \alpha \sup \ell(g_i \gamma) - \beta - \sup d(x, g_i x) \\ &\geq \alpha A \|\gamma\| - B\alpha - \beta - \sup d(x, g_i x). \end{aligned}$$

Also if $\gamma = \gamma_1 \cdots \gamma_k$ for γ_i in generating set,

$$d(x, \gamma x) \leq d(x, \gamma_k x) + d(\gamma_k x, \gamma_{k-1} \gamma_k x) + \cdots + d(\gamma_2 \cdots \gamma_k x, \gamma x) \leq C \|\gamma\|$$

for some C , which shows that the orbit map is a quasi-isometric embedding. Labourie [18] showed that Hitchin representations are well displacing. Hence the orbit map of any Hitchin representation is a quasi-isometric embedding.

5. LIMIT SET OF CONVEX REAL PROJECTIVE SURFACES

In this section we give an example of a limit set which is a topological circle in Furstenberg boundary. The example comes from a strictly convex real projective structure on a closed surface. The main source is from [15]. As far as we know, this is the first example of a concrete description of a limit set in higher rank symmetric space, which is quite interesting in its own right.

A real projective structure on a manifold M is a maximal atlas $\{U_i, \phi_i\}$ into \mathbb{RP}^d so that the transition functions $\phi_i \circ \phi_j^{-1}$ are restrictions of projective automorphisms of \mathbb{RP}^d . A (strictly) convex real projective surface S is Ω/Γ where Ω is a (strictly) convex domain in \mathbb{RP}^2 and Γ is a discrete subgroup of $\text{Aut}(\mathbb{RP}^2)$. Up to taking a subgroup of index two, we can assume that $\Gamma \subset \text{SL}(3, \mathbb{R})$.

An element $g \in \text{GL}(d, \mathbb{R})$ is called *proximal* if $\lambda_1(g) > \lambda_2(g)$ where $\lambda_1(g) \geq \lambda_2(g) \geq \dots \geq \lambda_d(g)$ is the sequence of modules of eigenvalues of g repeated with multiplicity. It is called *biproximal* if g^{-1} is also proximal. A proximal element is called *positively proximal* if the eigenvalue corresponding to $\lambda_1(g)$ is a positive real number. When $S = \Omega/\Gamma$ is a closed convex real projective surface with $\chi(S) < 0$, Kuiper [16] showed that Ω is strictly convex with $\partial\Omega$ at least C^1 , and every homotopically nontrivial closed curve on S is freely homotopic to a unique closed geodesic (in the Hilbert metric) which represents a positively biproximal element in $\text{SL}(3, \mathbb{R})$.

From now on, we fix $S = \Omega/\Gamma$ a strictly convex real projective closed surface such that any element in $\Gamma \subset \text{SL}(3, \mathbb{R})$ is positively biproximal and we set $X = \text{SL}(3, \mathbb{R})/\text{SO}(3)$. We will show that the limit set of Γ in the Furstenberg boundary of X is a circle. Note that by Benoist [3] if Ω is not an ellipsoid (in ellipsoid case Ω is a real hyperbolic 2-plane), Γ is Zariski dense in $\text{SL}(3, \mathbb{R})$ and the intersection of the limit set with a Weyl chamber at infinity, if nonempty, has nonempty interior by [4]. So the limit set itself cannot be homeomorphic to a circle. By this reason we consider limit set in the Furstenberg boundary of X . For a general connected semisimple Lie group with trivial center and no compact factors, the *Furstenberg boundary* is homeomorphic to G/P where P is a minimal parabolic subgroup. A *limit set* Λ_Γ of Γ in the *Furstenberg boundary* is the closure of the attracting fixed points of elements in Γ . See [3, Lemme 2.6].

Theorem 5.1. *Let Γ be a discrete subgroup of $\text{SL}(3, \mathbb{R})$ arising from a convex real projective structure on a closed surface. Then the limit set Λ_Γ of Γ in the Furstenberg boundary of $\text{SL}(3, \mathbb{R})/\text{SO}(3)$ is a circle.*

Proof. Let $S = \Omega/\Gamma$ be the convex real projective closed surface. For any $g \in \Gamma$, g has an attracting fixed point $\langle v^+ \rangle$ and a repelling fixed point $\langle v^- \rangle$ in $\partial\Omega$ corresponding to $\lambda_1(g)$ and $\lambda_3(g)$. Since g is biproximal, all the eigenvalues are positive reals and $\lambda_1(g) > \lambda_2(g) > \lambda_3(g)$.

If v_0 denotes the eigenvector corresponding to $\lambda_2(g)$, g fixes a flag

$$\langle v^+ \rangle \subset \langle v^+, v_0 \rangle \subset \mathbb{R}^3$$

with eigenvalues $\lambda_1(g), \lambda_2(g), \lambda_3(g)$. Also since g leaves invariant $\langle v^+, v_0 \rangle$ and $\langle v^-, v_0 \rangle$, $\langle v_0 \rangle$ is a unique intersection point of a line $\langle v^+, v_0 \rangle$ and a line $\langle v^-, v_0 \rangle$ in \mathbb{RP}^2 . Since these lines cannot pass through the interior of Ω (otherwise $\langle v_0 \rangle$ is on $\partial\Omega$, then g will have three fixed points on $\partial\Omega$, which is not allowed), these lines are tangent to $\partial\Omega$ at $\langle v^+ \rangle$ and $\langle v^- \rangle$ respectively, so $\langle v_0 \rangle$ is uniquely determined by $\langle v^+ \rangle$ and $\langle v^- \rangle$. Note that in this flag, $\langle v^+, v_0 \rangle$ is a line tangent to $\partial\Omega$ at $\langle v^+ \rangle$ in \mathbb{RP}^2 .

This eigenvalue-flag pair is a limit point of $g^{n>0}I$ in ∂X , which is a regular point in the Weyl chamber of ∂X corresponding to $\langle v^+ \rangle \subset \langle v^+, v_0 \rangle \subset \mathbb{R}^3$ since $\lambda_1(g) > \lambda_2(g) > \lambda_3(g)$, and also this flag is an attracting fixed point of g in the Furstenberg boundary of X . So for any $g \in \Gamma$, g determines a unique fixed point $\langle v^+ \rangle$ on $\partial\Omega$, and in turn this determines a unique flag $\langle v^+ \rangle \subset \langle v^+, v_0 \rangle \subset \mathbb{R}^3$ where $\langle v^+, v_0 \rangle$ is a line in \mathbb{RP}^2 tangent to $\partial\Omega$ at $\langle v^+ \rangle$, which is an attracting fixed point of g in the Furstenberg boundary of X .

Note that in this correspondence, for any $g \in \Gamma$, the attracting fixed point of g in Furstenberg boundary is a flag determined by $\langle v^+ \rangle \in \partial\Omega$ and a line through $\langle v^+ \rangle$ tangent to $\partial\Omega$. So a point in the closure of attracting fixed points of elements in Γ in the Furstenberg boundary, is determined by a tangent line through some point on $\partial\Omega$. But a point on $\partial\Omega$ determines a unique tangent line since Ω is strictly convex and $\partial\Omega$ is C^1 by Kuiper [16]. This shows that the limit set of Γ in the Furstenberg boundary, which is the closure of attracting fixed points of elements in Γ by [3], is homeomorphic to $\partial\Omega$ which is a circle. Note here that we can apply Benoist's theorem since Γ is Zariski dense either in $\mathrm{SL}(3, \mathbb{R})$ (if Ω is not an ellipse) or in $\mathrm{SO}(2, 1)$ (when Ω is an ellipse). \square

As observed already, the hyperbolic plane sits inside X as $\mathbb{R} \times \mathbb{H}^2$ where \mathbb{R} is a singular geodesic. Fix $x_0 \in \mathbb{H}^2$, then for any geodesic l through x_0 , $\mathbb{R} \times l$ is a maximal flat in X . Also $l(\infty)$ is a barycenter of a Weyl chamber so that $Td(\mathbb{R}(\infty), l(\infty)) = \pi/2$. If Γ is a Fuchsian group, then the geometric limit set is just a circle. Hence one can expect that if we perturb a Fuchsian group to a convex projective structure, then the geometric limit set would be a cylinder. Indeed,

by the result of [4, Section 7.5], one can show that the limit set L_Γ of Γ can be identified with $\Lambda_\Gamma \times \partial\mathcal{L}_\Gamma$ where \mathcal{L}_Γ is a limit cone. See also [7] and [19]. So the limit set L_Γ in ∂X is identified to a cylinder. More rigorously,

Theorem 5.2. *Let $S = \Omega/\Gamma$ be a compact strictly convex real projective surface. Then the limit set Λ_Γ of Γ in the Furstenberg boundary is homeomorphic to S^1 and the limit set L_Γ in the geometric boundary splits as a product $S^1 \times I$ where I is the closed interval. Furthermore for every pair of distinct points $p, q \in \partial\Omega$, the corresponding Weyl chambers W_p, W_q are opposite.*

Proof. By Benoist [4], for each $p \in \partial\Omega$, $W_p \cap L_\Gamma$ can be identified with the set of directions of the limit cone \mathcal{L}_Γ . The only obstruction for L_Γ to be a cylinder is that \mathcal{L}_Γ contains a singular direction and for two distinct $p, q \in \partial\Omega$, W_p, W_q are adjacent. For two distinct points $p, q \in \partial\Omega$, choose a sequence $\gamma_n \in \Gamma$, so that γ_n^+ and γ_n^- converge to p and q respectively. Since two lines $\langle \gamma_n^+, \gamma_n^0 \rangle$ and $\langle \gamma_n^-, \gamma_n^0 \rangle$ intersect at $\langle \gamma_n^0 \rangle$, and $\partial\Omega$ is C^1 , $\gamma_n^0 \rightarrow v_0$ in \mathbb{RP}^2 . Hence these lines converge to $\langle p, v_0 \rangle$, $\langle q, v_0 \rangle$. Two flags

$$\langle p \rangle \subset \langle p, v_0 \rangle \subset \mathbb{R}^3 \text{ and } \langle q \rangle \subset \langle q, v_0 \rangle \subset \mathbb{R}^3$$

correspond to two Weyl chambers W_p and W_q respectively. Since $p \neq q$, two Weyl chambers W_p and W_q are opposite due to equation (1). The same argument holds for any distinct pairs $p, q \in \partial\Omega$. This shows that Weyl chambers corresponding to two distinct pairs are opposite, and consequently they are not adjacent. Hence the geometric limit set L_Γ is homeomorphic to $\Lambda_\Gamma \times \partial\mathcal{L}_\Gamma$. \square

Indeed, the first statement in Theorem 5.2 easily follows from the result of Sambarino [21] that the limit cone of any discrete group in the Hitchin component of $\mathrm{SL}(d, \mathbb{R})$ is contained in the interior of the Weyl chamber. This implies that every limit point in L_Γ is regular.

6. CHARACTERISATION OF LIMIT POINTS IN CONVEX REAL PROJECTIVE SURFACES

Let K be an infinite compact metrisable topological space. Suppose that a group G acts by homeomorphism on K . A group G is said to be a *convergence group* if the induced action on the space of distinct triples

of K is properly discontinuous, or equivalently if a given sequence of distinct $g_i \in G$, there are points c and b of K and a subsequence (g_{n_i}) such that

$$g_{n_i}(z) \rightarrow b$$

uniformly outside neighborhoods of c . Convergence groups acting on the standard sphere or ball of \mathbb{R}^n were first introduced by Gehring and Martin [10]. Then Freden [9] and Tukia [22] generalized the notion of convergence group to groups acting on spaces other than the sphere or the ball and having the convergence property. For instance, a group of isometries of a Gromov hyperbolic space can be extended to the Gromov boundary as a convergence group [22]. For further discussion, see [5], [9], [23].

The limit set L_G of G is the set of limit points, where a limit point is an accumulation point of some G -orbit in K . The limit set is the unique minimal closed nonempty G -invariant subset of K and G acts properly discontinuously on the $K \setminus L_G$. A point $z \in L_G$ is said to be a *conical limit point* if there is a sequence (g_n) of distinct elements of G such that, for every $x \in L_G \setminus \{z\}$, the sequence $(g_n x, g_n z)$ is relatively compact in $L_G \times L_G \setminus \Delta_G$ where $\Delta_G = \{(y, y) \mid y \in L_G\}$.

Let Γ be a closed surface group and C_Γ be the Caley graph of Γ . Since Γ is a hyperbolic group, the Gromov boundary ∂C_Γ of Γ is well defined up to Hölder homeomorphism. Hence the group Γ is a convergence group acting by homeomorphism on the Gromov boundary. Furthermore, it is well known that $L_\Gamma = \partial C_\Gamma$ and every point of ∂C_Γ is a conical limit point [22].

Let $S = \Omega/\Gamma$ be a strictly convex real projective closed surface for $\Gamma \subset \mathrm{SL}(3, \mathbb{R})$. Since Γ is a closed surface group and acts cocompactly on Ω , there is a canonical identification of ∂C_Γ with $\partial\Omega$. Hence, it is obvious that every point of $\partial\Omega$ is a conical limit point with respect to the action of Γ on $\partial\Omega$.

Recall that the Furstenberg boundary $\partial_F X$ of X can be identified with the set of equivalence classes of Weyl chambers in maximal flats in X . Here, two Weyl chambers W_1, W_2 are called *equivalent* if

$$d_H(W_1, W_2) < +\infty,$$

where d_H is the Hausdorff distance on subsets of X . For each chamber W , denote its equivalence class by $[W]$. The usual angle at a point x in X subtended by the vectors of the centers of gravity of Weyl chambers

in the unit tangent space $S_x X$ gives rise to a metric in $\partial_F X$. For more details, see [8, Section 3.8].

Let $\langle v \rangle$ be a point in $\partial\Omega \subset \mathbb{RP}^2$. As we observed in the proof of Theorem 5.1, $\langle v \rangle$ uniquely determines the flag $\langle v \rangle \subset \langle v, v_0 \rangle \subset \mathbb{R}^3$ where $\langle v, v_0 \rangle$ is the 2-dimensional plane corresponding to the unique tangent line to $\partial\Omega$ at $\langle v \rangle$. This flag determines a Weyl chamber, denoted by $W_{\langle v \rangle}$. Now define a map $\phi : \partial\Omega \rightarrow \partial_F X$ by

$$\phi(\langle v \rangle) = [W_{\langle v \rangle}]$$

for $\langle v \rangle \in \Omega$. It can be easily seen that this map is a Γ -equivariant homeomorphism onto its image $\Lambda_\Gamma = \phi(\partial\Omega)$. Due to this Γ -equivariant homeomorphism $\phi : \partial\Omega \rightarrow \Lambda_\Gamma$, every point of Λ_Γ is a conical limit point with respect to the action of Γ on Λ_Γ .

Proposition 6.1. *Let Γ be a discrete subgroup of $\mathrm{SL}(3, \mathbb{R})$ arising from a convex projective structure on a closed surface. Every Weyl chamber at infinity with nonempty limit set of Γ in the geometric boundary of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ contains at least one radial limit point.*

Proof. Due to the homeomorphism $\phi : \partial\Omega \rightarrow \Lambda_\Gamma$, every point in Λ_Γ is of the form $[W_p]$ for some $p \in \partial\Omega$. It is sufficient to prove that the Weyl chamber W_p at infinity contains a radial limit point.

As observed before, every point in Λ_Γ is a conical limit point. Hence there is a sequence (γ_n) of distinct elements of Γ such that for every $[W_q] \in \Lambda_\Gamma \setminus \{[W_p]\}$, the sequence $(\gamma_n[W_p], \gamma_n[W_q])$ is relatively compact in $\Lambda_\Gamma \times \Lambda_\Gamma \setminus \Delta_\Gamma$ where Δ_Γ is the diagonal in $\Lambda_\Gamma \times \Lambda_\Gamma$. Thus, by passing to a subsequence, we can assume that $\gamma_n[W_p]$ converges to $[W_a]$ and $\gamma_n[W_q]$ converges to $[W_b]$ for some distinct points $a, b \in \partial\Omega$.

For a Weyl chamber W , write $W(\infty) = \overline{W} \cap \partial X$ where \overline{W} is the closure in the compactification $X \cup \partial X$ of X . Note that if two chambers W_1 and W_2 are equivalent, $W_1(\infty) = W_2(\infty)$. Thus $[W](\infty) = W(\infty)$ is well defined for each equivalence class $[W]$. It is a standard fact that if two Weyl chambers in the Furstenberg boundary are opposite, there exists a unique maximal flat connecting them. Since any two distinct Weyl chambers in Λ_Γ are opposite by Theorem 5.2, there is a unique maximal flat connecting $\gamma_n[W_p](\infty)$ and $\gamma_n[W_q](\infty)$. Let F_n denote such maximal flat. Denote by F_0 (resp. F) the unique maximal flat connecting $[W_p](\infty)$ (resp. $[W_a](\infty)$) and $[W_q](\infty)$ (resp. $[W_b](\infty)$). Then, it is obvious that $F_n = \gamma_n F_0$.

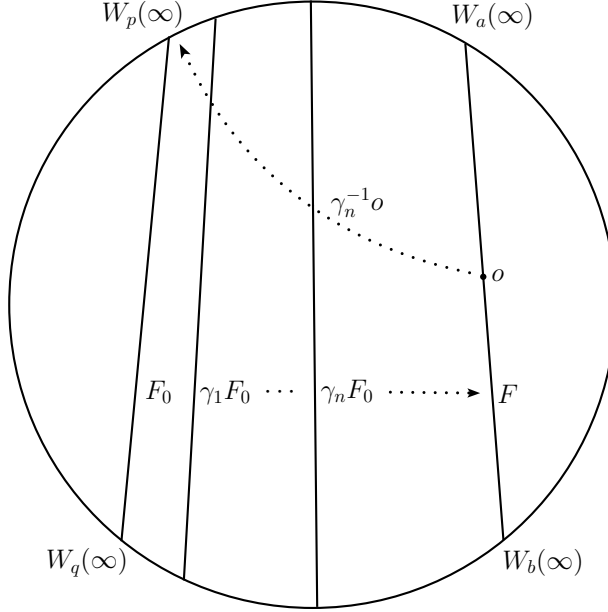


FIGURE 1. Limit set and Weyl chambers.

Since $\gamma_n[W_p](\infty)$ and $\gamma_n[W_q](\infty)$ converge to $[W_a](\infty)$ and $[W_b](\infty)$ respectively, F_n should converge to F . More precisely, there is a sequence $(o_n) \in F_n$ and $o \in F$ such that (o_n, F_n) converges to (o, F) in the space of pointed flats. This implies that for any $C > 0$ there exists $N > 0$ such that

$$B(o, R) \cap F_n \neq \emptyset$$

for all $n \geq N$. Thus we have $d(o, F_n) = d(o, \gamma_n F_0) = d(\gamma_n^{-1} o, F_0) < C$. In other words, the sequence $(\gamma_n^{-1} o)$ remains at a bounded distance C of the maximal flat F_0 . By the discreteness of Γ , $(\gamma_n^{-1} o)$ can not accumulate to a point in X . Hence it should converge to a boundary point in ∂X and moreover, the boundary point is in $F_0(\infty)$ due to $d(\gamma_n^{-1} o, F_0) < C$ for all sufficiently large n .

Suppose that $(\gamma_n^{-1} o)$ converges to a point $z \in F_0(\infty)$. Because z is an accumulation point of the Γ -orbit of the point $o \in X$, it should be in the limit set L_Γ . According to Theorem 5.2, W_p and W_q are opposite in F_0 and z should be in either $W_p(\infty)$ or $W_q(\infty)$. Noting

that all arguments above hold for any $q \in \partial\Omega \setminus \{p\}$, one can easily see that z should be in $W_p(\infty)$. See the Figure 1.

Let $Pr : X \rightarrow F_0$ be the orthogonal projection onto F_0 . Since the sequence $(\gamma_n^{-1}o)$ remains at a bounded distance C of the maximal flat F_0 , we have

$$d(\gamma_n^{-1}o, Pr(\gamma_n^{-1}o)) < C.$$

This implies that the sequence $(Pr(\gamma_n^{-1}o))$ also converges to z . Since every limit point in $[W_p](\infty)$ is regular as we mentioned before, the sequence $(Pr(\gamma_n^{-1}o))$ lies in W_p^0 for all large n where W_p^0 is a Weyl chamber in F_0 representing $[W_p]$. This implies

$$d(\gamma_n^{-1}o, W_p^0) = d(\gamma_n^{-1}o, Pr(\gamma_n^{-1}o)) < C.$$

Finally, we can conclude that z is a radial limit point in $W_p(\infty)$. This completes the proof. \square

Theorem 6.2. *Let Γ be a discrete subgroup of $SL(3, \mathbb{R})$ arising from a convex projective structure on a closed surface. Then its limit set in the geometric boundary of $SL(3, \mathbb{R})/SO(3)$ consists of only horospherical limit points.*

Proof. We stick to the notation in the proof of Proposition 6.1. Let w be a limit point in $W_p(\infty)$ and z a radial limit point in $W_p(\infty)$. Let \mathcal{H} be a horoball based at w . The Tits distance $Td(z, w)$ is less than $\pi/3$. Let σ_1 be the geodesic ray emanating from a point of F_0 and tending to z . Let σ_2 be a geodesic such that

$$\sigma_2(\infty) = w, \quad \mathcal{H} = b(\sigma_2)^{-1}((-\infty, 0)),$$

where $b(\sigma_2) : X \rightarrow \mathbb{R}$ is the Busemann function associated with σ_2 . To prove the theorem, we start by observing the following lemma.

Lemma 6.3. *Let F be a maximal flat in X and (x_n) be a sequence of points in F that converges to z in $F(\infty)$. Let $\sigma : [0, \infty) \rightarrow X$ be a geodesic ray in F tending to z . Then, for any $\epsilon > 0$, there exist a sequence (t_n) in $[0, \infty)$ and $N > 0$ such that*

$$d(\sigma(t_n), x_n) < \epsilon t_n,$$

for all $n \geq N$.

Proof. Let $\sigma(t_n)$ be the projection point of x_n onto the geodesic ray σ . Denote by $\angle_{\sigma(0)}(x_n, \sigma(t_n))$ the angle subtended at $\sigma(0)$ by x_n and

$\sigma(t_n)$. Then, $\angle_{\sigma(0)}(x_n, \sigma(t_n))$ converges to zero by the definition of the cone topology on $X \cup \partial X$. Hence we have

$$\tan \angle_{\sigma(0)}(x_n, \sigma(t_n)) = \frac{d(\sigma(t_n), x_n)}{t_n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, for a given $\epsilon > 0$, there exists $N > 0$ such that

$$\frac{d(\sigma(t_n), x_n)}{t_n} < \epsilon.$$

for all $n \geq N$. Therefore $d(\sigma(t_n), x_n) < \epsilon t_n$ for all $n \geq N$. \square

According to Lemma 6.3, for any $\epsilon > 0$, there exist a sequence (t_n) in $[0, \infty)$ and $N > 0$ such that

$$d(\sigma_1(t_n), Pr(\gamma_n^{-1}o)) < \epsilon t_n,$$

for all $n \geq N$. Choose an $\epsilon < \cos(Td(z, w))$. Since the norms of gradient vectors of the Busemann function $b(\sigma_2)$ are equal to 1, we have

$$|b(\sigma_2)(\gamma_n^{-1}o) - b(\sigma_2)(Pr(\gamma_n^{-1}o))| \leq d(\gamma_n^{-1}o, Pr(\gamma_n^{-1}o)) < C.$$

In the same way, we obtain

$$|b(\sigma_2)(\sigma_1(t_n)) - b(\sigma_2)(Pr(\gamma_n^{-1}o))| \leq d(\sigma_1(t_n), Pr(\gamma_n^{-1}o)) < \epsilon t_n,$$

for all $n \geq N$. Furthermore, since $Td(z, w) < \pi/3$, it follows from [13, Lemma 3.4] that

$$b(\sigma_2)(\sigma_1(t)) < -t \cdot \cos(Td(z, w)) + D,$$

for some $D > 0$. Then for all $n \geq N$,

$$\begin{aligned} b(\sigma_2)(\gamma_n^{-1}o) &= b(\sigma_2)(\gamma_n^{-1}o) - b(\sigma_2)(Pr(\gamma_n^{-1}o)) \\ &\quad + b(\sigma_2)(Pr(\gamma_n^{-1}o)) - b(\sigma_2)(\sigma_1(t_n)) + b(\sigma_2)(\sigma_1(t_n)) \\ &< C + \epsilon t_n - t_n \cdot \cos(Td(z, w)) + D \\ &< C - t_n(1/2 - \epsilon) + D. \end{aligned}$$

Since the sequence (t_n) goes to infinity, one can choose a sufficiently large $N > 0$ such that for all $n \geq N$,

$$b(\sigma_2)(\gamma_n^{-1}o) < 0.$$

Hence $\gamma_n^{-1}o \in \mathcal{H}$ for all $n \geq N$. Therefore, we can conclude that w is a horospherical limit point. \square

7. FURTHER CHARACTERIZATION OF LIMIT POINTS IN CONVEX REAL PROJECTIVE SURFACES

In the previous section, we prove that every Weyl chamber at infinity with nonempty limit set of a discrete group Γ in the Hitchin component for $\mathrm{SL}(3, \mathbb{R})$ has at least one radial limit point. One can ask how many limit points in each Weyl chamber at infinity are radial limit points. In this section, we answer this question and describe where the set of radial limit points is positioned in the limit set of Γ .

Theorem 7.1. *Let Γ be a discrete subgroup of $\mathrm{SL}(3, \mathbb{R})$ arising from a convex projective structure on a closed surface. Then there is only one radial limit point in any Weyl chamber at infinity with nonempty limit set of Γ in the geometric boundary of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$.*

Proof. Let W be a Weyl chamber in $X = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ such that the limit set of Γ in $W(\infty)$ is nonempty. As we observed before, we can assume $W = \phi(p) = W_p$ for some $p \in \partial\Omega$. Suppose that z is a radial limit point in $W_p(\infty)$. Fix a point $o \in X$. By the definition of radial limit point, there exists a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of Γ and a constant $C > 0$ such that

$$(2) \quad d(\gamma_n o, W_p) < C$$

for all $n \in \mathbb{N}$.

Choose a point $q \in \partial\Omega$ distinct from p . Since $W_p(\infty)$ and $W_q(\infty)$ are opposite, there exists a unique maximal flat F_0 connecting $W_p(\infty)$ and $W_q(\infty)$. Inequality (2) implies that for some $C_0 > 0$,

$$d(\gamma_n o, F_0) < C_0.$$

In other words, $d(o, \gamma_n^{-1} F_0) < C_0$. Since the sequence $(\gamma_n^{-1} F_0)$ of maximal flats remains at a bounded distance of a point $o \in X$, it converges to a maximal flat F in the space of pointed maximal flats (See [8, Section 8.3 and 8.4]). Then the sequences $\gamma_n^{-1} W_p(\infty)$ and $\gamma_n^{-1} W_q(\infty)$ converge to $W_a(\infty)$ and $W_b(\infty)$ in $F(\infty)$ respectively for some $a, b \in \partial\Omega$. Furthermore, since $\gamma_n^{-1} W_p(\infty)$ and $\gamma_n^{-1} W_q(\infty)$ are opposite, $W_a(\infty)$ and $W_b(\infty)$ should be opposite. This implies $p \neq q$ and thus, there exists a unique geodesic joining a and b in Ω .

On the other hand, due to the Γ -equivariant map $\phi : \partial\Omega \rightarrow \Lambda_\Gamma$, the sequences $(\gamma_n^{-1} p)$ and $(\gamma_n^{-1} q)$ converge to a and b respectively. Let l_{pq} be the geodesic connecting p and q with respect to the Hilbert metric

on Ω . Then the sequence $(\gamma_n^{-1}l_{pq})$ converges to l_{ab} and thus, for a point $e \in l_{ab}$, there is a constant $D > 0$ such that for all large n ,

$$d(\gamma_n e, l_{pq}) = d(e, \gamma_n^{-1}l_{pq}) < D.$$

Hence, the sequence $(\gamma_n e)$ should converge to either p or q . Noting that all arguments above hold for any $q \in \Omega \setminus \{p\}$, it can be easily seen that $(\gamma_n e)$ converges to p . Furthermore, since the sequence $(\gamma_n e)$ remains at a bounded distance of l_{pq} , the broken geodesic ray consisting of geodesic segments $[\gamma_n e, \gamma_{n+1} e]$ becomes a quasi-geodesic ray in Ω by choosing its subsequence so that the distance between any two points is greater than a sufficiently large constant.

Now, let's consider the broken geodesic ray \mathcal{R} in the Cayley graph C_Γ consisting of geodesic segments $[\gamma_n, \gamma_{n+1}]$. Then the ray \mathcal{R} is also a quasi-geodesic ray in the Cayley graph C_Γ because C_Γ and Ω are quasi-isometric by an orbit map. Via a canonical identification between ∂C_Γ and $\partial \Omega$, we can assume that the sequence γ_n converges to $p \in \partial C_\Gamma$. According to the Morse lemma, the quasi-geodesic ray \mathcal{R} remains at a bounded Hausdorff distance of a geodesic ray in C_Γ joining id and p where id is the identity element of Γ .

Suppose that z' is another radial limit point in $W_p(\infty)$ and $\gamma'_n o$ converges to z' with $d(\gamma'_n o, W_p) < C'$ for some $C' > 0$. In the same way as above, we get a quasi-geodesic ray \mathcal{R}' in C_Γ consisting of geodesic segments $[\gamma'_n, \gamma'_{n+1}]$ whose endpoint is $p \in \partial C_\Gamma$. Noting that \mathcal{R} and \mathcal{R}' are quasi-geodesic rays in C_Γ with the same endpoint, it can be easily seen by the Morse lemma that \mathcal{R}' remains at a bounded Hausdorff distance of \mathcal{R} . Moreover, since the orbit map $C_\Gamma \rightarrow X$ is a quasi-isometric embedding (see section 4), two quasi-geodesic rays $\mathcal{R}o$ and $\mathcal{R}'o$ in X should have the same endpoint at infinity. This means that $z = z'$. Therefore, $W_p(\infty)$ contains exactly one radial limit point. \square

Theorem 5.2 and 7.1 imply that the set of radial limit points of Γ is isomorphic to a circle S^1 in the category of sets. Moreover, Link [20] proved that if Γ is a non-elementary discrete group, then the set of attracting fixed points of regular axial isometries is a dense subset of the limit set L_Γ . Since the set of radial limit points contains the set of attractive fixed points of regular axial isometries, the set of radial limit points is also dense in L_Γ . Hence we have the following immediate corollary.

Corollary 7.2. *Let Γ be a discrete subgroup of $\mathrm{SL}(3, \mathbb{R})$ arising from a convex projective structure on a closed surface. Then the set of radial limit points of Γ in the geometric boundary of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ is isomorphic to a circle S^1 and dense in the limit set L_Γ of Γ .*

8. ANOSOV REPRESENTATIONS IN SEMISIMPLE LIE GROUP

A Fuchsian representation from $\pi_1(S)$ to $\mathrm{PSL}(n, \mathbb{R})$, where S is a closed surface with genus ≥ 2 , is a representation $\rho = \iota \circ \rho_0$, where ρ_0 is a Fuchsian representation in $\mathrm{PSL}(2, \mathbb{R})$ and ι is the irreducible representation of $\mathrm{PSL}(2, \mathbb{R})$ in $\mathrm{PSL}(n, \mathbb{R})$. A Hitchin component is the connected component of a representation variety which contains fuchsian representations. In [17], it is shown that a Hitchin representation is hyperconvex and vice versa in the following sense; a representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}(n, \mathbb{R})$ is hyperconvex if there exists a ρ -equivariant hyperconvex curve ξ from $\partial_\infty \pi_1(S)$ in \mathbb{RP}^{n-1} , i.e., for any distinct points (x_1, \dots, x_n) , the sum $\xi(x_1) + \dots + \xi(x_n)$ is direct. Such ξ is unique and ξ is called the limit curve of ρ . In [17], it is shown that a Hitchin representation ρ is hyperconvex and discrete, faithful. Furthermore, $\rho(\gamma)$, $id \neq \gamma \in \pi_1(S)$ is real split with distinct eigenvalues. If γ^+ is the attracting fixed point of γ in $\partial_\infty \pi_1(S)$, then $\xi(\gamma^+)$ is the unique attracting fixed point of $\rho(\gamma)$ in \mathbb{RP}^{n-1} . Furthermore the limit curve ξ is a hyperconvex Frenet curve: there exists a family $(\xi = \xi^1, \xi^2, \dots, \xi^{n-1})$ called the osculating flag so that

- (1) ξ^p takes values in the Grassmannian of p -planes,
- (2) $\xi^p(x) \subset \xi^{p+1}(x)$,
- (3) if (n_1, \dots, n_l) are positive integers such that $\sum n_i \leq n$ and if (x_1, \dots, x_l) are distinct points, the sum

$$\xi^{n_1}(x_1) + \dots + \xi^{n_l}(x_l)$$

is direct.

- (4) If $p = n_1 + \dots + n_l$, then for all distinct points (y_1, \dots, y_l) ,

$$\lim_{(y_1, \dots, y_l) \rightarrow x} \oplus \xi^{n_i}(y_i) = \xi^p(x).$$

Specially if we take $x \neq y$ on $\partial \pi_1(S)$, then for any $n_1 + n_2 = n$, $\xi^{n_1}(x) + \xi^{n_2}(y) = \mathbb{R}^n$. Hence for such a representation in a Hitchin component, any two distinct points on the ideal boundary define two opposite Weyl chambers in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$. Then all the previous

arguments work in this case also. Hence Theorem 1.5 immediately follows.

In [12], this notion is generalized to the finitely generated word hyperbolic group Γ as follows. A representation $\rho : \Gamma \rightarrow G$ to a semisimple Lie group G is P^+ -Anosov if there exist continuous ρ -equivariant maps $\psi^+ : \partial\Gamma \rightarrow G/P^+$ and $\psi^- : \partial\Gamma \rightarrow G/P^-$, where P^\pm are two opposite parabolic subgroups, such that

- (1) for all $(t, t') \in \partial\Gamma \times \partial\Gamma \setminus \Delta$, the pair $(\psi^+(t), \psi^-(t'))$ is in the unique open G -orbit in $G/P^+ \times G/P^-$. Here Δ is a diagonal set.
- (2) for all $t \in \partial\Gamma$, the pair $(\psi^+(t), \psi^-(t))$ is contained in a unique closed G -orbit in $G/P^+ \times G/P^-$.
- (3) they satisfy some contraction property with respect to the flow.

Two such examples are;

- (1) Let G be a split real simple Lie group and S be a closed oriented surface of genus ≥ 2 . Representations $\rho : \pi_1(S) \rightarrow G$ in the Hitchin component are B -Anosov where $B \subset G$ is a Borel subgroup.
- (2) The holonomy representation $\rho : \pi_1(M) \rightarrow \mathrm{PGL}(n+1, \mathbb{R})$ of a strictly convex real projective structure on an n -dimensional manifold M is P -Anosov where $P \subset \mathrm{PGL}(n+1, \mathbb{R})$ is the stabilizer of a line.

Hence if $\rho : \Gamma \rightarrow G$ is a P -Anosov representation from a word hyperbolic group Γ where P is a minimal parabolic subgroup of a semisimple Lie group G . In this case $P = P^+ = MAN$ and its opposite minimal parabolic subgroup $P^- = MAN^-$ are conjugate. Hence two spaces G/P and G/P^- are canonically identified with the set of Weyl chambers at infinity. Then by the first property of the maps ψ^\pm , for any distinct elements $t, t' \in \partial\Gamma$, $\psi^+(t)$ and $\psi^-(t')$ are opposite Weyl chambers of the symmetric space G/K . Specially by the uniqueness of ψ^\pm , $\psi^+ = \psi^-$ (see section 4.5 in [12]), and hence we obtain

Lemma 8.1. *The map ψ^+ is injective and for any distinct elements $t, t' \in \partial\Gamma$, $\psi^+(t)$ and $\psi^+(t')$ are opposite Weyl chambers of the symmetric space G/K . Furthermore the orbit map $\Gamma \rightarrow G/K$ is a quasi-isometric embedding.*

Proof. If $\psi^+(t) = \psi^+(t')$ for two distinct elements t and t' , since $\psi^+ = \psi^-$ it will contradict the fact that $\psi^+(t)$ and $\psi^-(t')$ are opposite. The second statement follows from the property (ii) of Theorem 1.7 in [12]. \square

Therefore all the previous arguments work in this more general context as well.

Theorem 8.2. *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense discrete P -Anosov representation from a word hyperbolic group Γ where P is a minimal parabolic subgroup of a semisimple Lie group G . Then the geometric limit set is isomorphic to the set $\partial\Gamma \times \partial\mathcal{L}_\rho(\Gamma)$. Furthermore in each Weyl chamber intersecting the geometric limit set nontrivially, there is only one radial limit point.*

REFERENCES

- [1] P. Goddard, A. Kent and D. I. Olive, ‘Unitary representations of the Virasoro and Supervirasoro algebras’, *Comm. Math. Phys.* 103 (1986) 105.
- [2] P. Albuquerque, ‘Patterson-Sullivan theory in higher rank symmetric spaces’, *Geom. Funct. Anal.* 8 (1999) 1–28.
- [3] Y. Benoist, ‘Automorphismes des cônes convexes’, *Invent. Math.* 141 (1) (2000) 149–193.
- [4] Y. Benoist, ‘Propriété asymptotiques des groupes linéaires’, *Geom. Funct. Anal.* 7 (1997) 1–47.
- [5] B. H. Bowditch, ‘Convergence groups and configuration spaces’, in : “Groups theory down under” (J. Cossey, C. F. Miller, W. D. Neumann, and M. Shapiro, eds.), de Gruyter (1999) 23–54.
- [6] S. Choi and W. M. Goldman, ‘Convex real projective structures on closed surfaces are closed’, *Pro. Amer. Math. Soc.* vol. 118, no. 2 (1993) 657–661.
- [7] F. Dal’Bo and I. Kim, ‘Shadow lemma on the product of Hadamard manifolds and applications’, *Actes du séminaire de Théorie spectrale et géométrie* vol. 25 (Année 2006-2007) 105–120, Institut Fourier-Université de Grenoble I.
- [8] P. Eberlein, ‘Geometry of nonpositively curved manifolds’, University of Chicago Press (1996).
- [9] E. Freden, ‘Negatively curved groups have the convergence property I’. *Ann. Acad. Sci. Fenn. Ser. A I* 20 (1995) 333–348.
- [10] F. W. Gehring and G. J. Martin, ‘Discrete quasiconformal groups I’, *Proc. London Math. Soc.* (3) 55 (1987) 331–358.
- [11] T. Delzant, O. Guichard, F. Labourie and S. Mozes, ‘Well displacing representations and orbit maps’, to appear in Proceedings of Robert Zimmer’s birthday conference.

- [12] O. Guichard and A. Wienhard, ‘Anosov representations: Domain of discontinuity and applications’, to appear in *Inventiones Math.*
- [13] T. Hattori, ‘Geometric limit sets of higher rank lattices’, *Proc. London Math. Soc.* vol. 90 (3) (2005) 689–710.
- [14] S. Helgason, ‘Differential Geometry, Lie Groups, and Symmetric spaces’, Academic Press, (1978).
- [15] I. Kim, ‘Rigidity and deformation spaces of strictly convex real projective structures on compact manifolds’, *J. Diff. Geom.* vol. 58 (2) (2001) 189–218.
- [16] N. Kuiper, ‘On convex locally projective spaces’, *Convegno Int. Geometria Diff., Italy* (1954) 200–213.
- [17] F. Labourie, ‘Anosov flows, Surface groups and Curves in projective spaces’, *Invent. Math.* 165 (2006) no. 1, 51–114.
- [18] F. Labourie, ‘Cross ratio, Anosov representations and the energy on Teichmüller space’, *Ann. Sci. Éc. Norm. Supér.* (4) 41 (2008) no. 3, 437–469.
- [19] G. Link, ‘Limit sets of discrete groups acting on symmetric spaces’, (2002) Thesis, Karlsruhe.
- [20] G. Link, ‘Geometry and dynamics of discrete isometry groups of higher rank symmetric spaces’, *Geom. Dedicata* 122 (2006) 51–57.
- [21] A. Sambarino, ‘Hyperconvex representations and exponential growth’, arXiv:1203.0272.
- [22] P. Tukia, ‘Convergence groups and Gromov’s metric hyperbolic spaces’, *New Zealand J. Math.* 23 (1994) 157–187.
- [23] P. Tukia, ‘Conical limit points and uniform convergence groups’, *J. reine angew. math.* 501 (1998) 71–98.

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